Logical Characterizations of Weighted Complexity Classes

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SO logic

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Fagin's Theorem

[Fagin '73]

ESO = NP

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Immerman-Vardi's Theorem

[Immerman '86, Vardi '82]

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$\mathcal{A} \models \beta \Leftrightarrow M(\mathcal{A}) = 1$

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Example $(\mathbb{N}_0, +, \cdot, 0, 1)$

$[\![\bigoplus x. \bigoplus y.\text{edge}(x, y)]\!]$ = number of edges wSO[S] logic

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wESO[S] = NP[S] $\qquad [\![\varphi]\!](\mathcal{A}) = [\![M]\!](\mathcal{A})$

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Turing machine M over S

transitions Δ $\qquad$ wt : $\Delta \to S$

multiply along computation $\qquad$ sum over computations

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NP[$\mathbb{B}$] = NP

NP[$(\mathbb{N}_0, +, \cdot, 0, 1)$] = #P $\qquad M\colon \mathcal{A} \to |\mathrm{Run}_M(\mathcal{A})|$ $\qquad$ [Saluja et al., Arenas et al.]

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FP[$\mathbb{B}$] = P

FP[$(\mathbb{N}_0, +, \cdot, 0, 1)$] = FP $\qquad f$: structures $\to \mathbb{N}_0$ computable in P $\qquad$ [Arenas et al. '20]