Weighted Automata and Weighted Logics

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Weighted Automata

$$q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_2 \xrightarrow{b} q_3 \xrightarrow{a} q_4 \xrightarrow{b} q_5 \xrightarrow{a} q_6$$

Weighted Automata

Weights in $(S, \oplus, \odot, \mathbb{0}, \mathbb{1})$ [Schützenberger '61]

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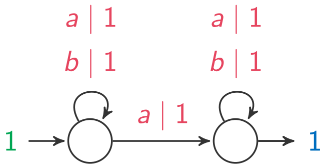

$w \mapsto |w|_a$

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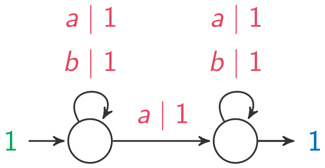

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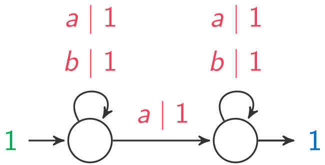

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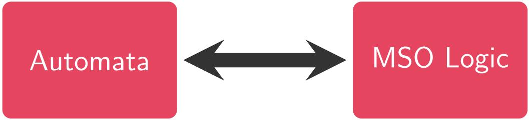
Books

Eilenberg	'74
Salomaa, Soittola	'78
Kuich, Salomaa	'86
Berstel, Reutenauer	'88
Droste, Kuich, Vogler	'09
Sakarovitch	'09

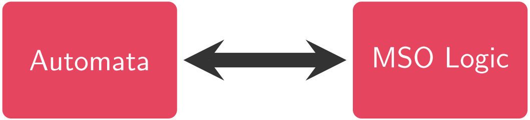
Büchi-Elgot-Trakhtenbrot '60/'61



Motivation

Büchi-Elgot-Trakhtenbrot '60/'61

$$\beta ::= P_a(x) \mid x \leq y \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$$



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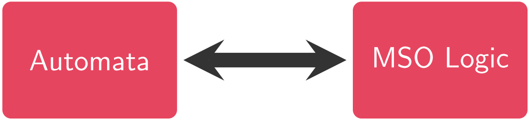

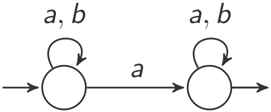

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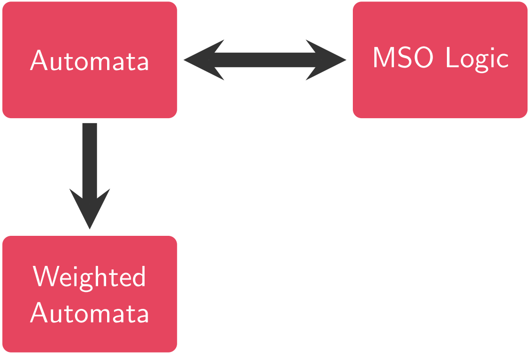
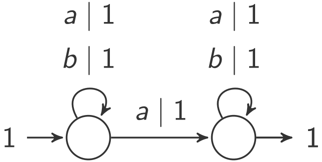


$$(\mathbb{N}_0, +, \cdot, 0, 1)$$



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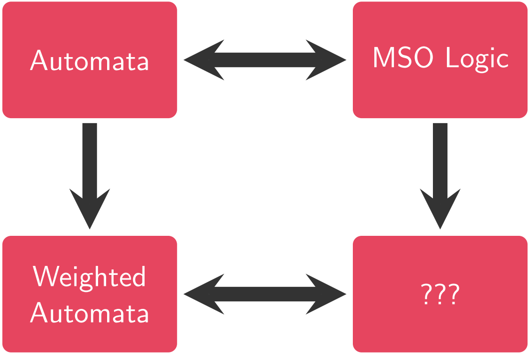

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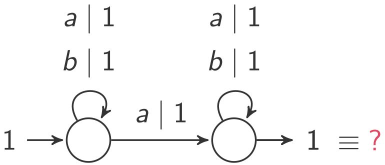

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[Droste, Gastin '05]

First Solution: Same syntax different semantics

$\beta ::= P_a(x) \mid x \leq y \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta \mid s$

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$s \in (S, \oplus, \odot, \mathbb{0}, \mathbb{1})$ $\quad\Rightarrow$ semantics of $\neg s$?

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Semantics: mimic $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$

- Atomic formulas: $\mathbb{0}$ or $\mathbb{1}$
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$$[\![\exists x.P_a(x)]\!](w) \equiv |w|_a$$

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fix order or assume commutativity

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restricted wMSO

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Thm restricted wMSO $\equiv$ weighted automata [Droste, Gastin '05]

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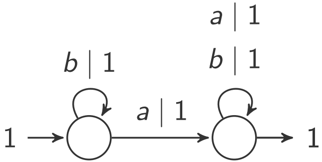

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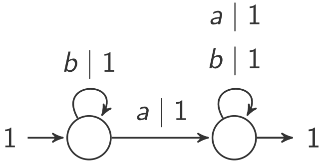

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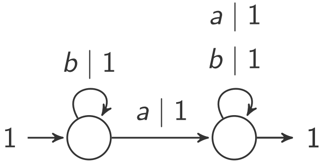

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$$\exists x.\left(P_a(x) \wedge \forall y\left(P_a(y) \wedge x \leq y \vee P_b(y)\right)\right)$$

$x =$ first a

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- syntactic restriction for $\forall x.\beta$
- shorter formulas

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$$[\![\exists x.P_a(x)]\!](w) = \mathbb{1}_{\Sigma^* a \Sigma^*}(w) \qquad [\![\bigoplus x.P_a(x)]\!](w) = |w|_a$$

Alternative

$\beta ::= P_a(x) \mid x \leq y \mid x \in X \mid \neg\beta \mid \beta \odot \beta \mid \bigodot x.\beta \mid \bigodot X.\beta$ $\mathbb{1}_{\Sigma^* a \Sigma^*} \equiv \neg \bigodot x.\neg P_a(x)$

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$+ \quad \max \quad + \quad \max \; + \quad \max \quad +$

$\bigodot x.(0 \oplus P_a(x) \odot 1) \;\oplus\; \bigodot x.(0 \oplus P_b(x) \odot 1)$ $\quad$ $w \mapsto \max\{|w|_a, |w|_b\}$

Beyond Words

Thm restricted wMSO $\equiv$ weighted automata

- Finite words [Droste, Gastin '05]
- Finite ranked trees [Droste, Vogler '06]
- Pictures [Fichtner '06][Babari '15]
- Infinite words [Droste, Rahonis '06]
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- Data words (weighted register automata) [Babari, Droste, Perevoshchikov '16]
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restrictions on $\bigodot x.\varphi$ and $\bigodot X.\varphi$ motivated?

Feferman-Vaught Theorem

formula β $\xleftrightarrow{\text{satisfaction}}$ structure $\mathcal{A}$

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Given $\beta \in$ MSO

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such that for all structures $\mathcal{A}$,$\mathcal{B}$

$\mathcal{A} \sqcup \mathcal{B} \models \beta$ iff $P(x_1, \ldots, x_n, y_1, \ldots, y_n) = \text{true}$

where $x_i = \text{true}$ iff $\beta_i^{1} \models \mathcal{A}$

$y_i = \text{true}$ iff $\beta_i^{2} \models \mathcal{B}$

$$\beta ::= R(x_1, \ldots, x_n) \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$$ MSO

$$P ::= x_i \mid y_i \mid P \vee P \mid P \wedge P$$ Propositional formulas Prop

$$\psi ::= \beta \mid s \mid \psi \oplus \psi \mid \psi \odot \psi$$

$$\varphi ::= \beta \mid s \mid \varphi \oplus \varphi \mid \varphi \odot \varphi \mid \bigoplus x.\varphi \mid \bigodot x.\psi \mid \bigoplus X.\varphi$$ wMSO

$$E ::= x_i \mid y_i \mid E \oplus E \mid E \odot E$$ Expressions

Given $\beta \in \text{MSO}$

there exist $n \geq 1$ $\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{\mathbb{2}} \in \text{MSO}^n$ $P \in \text{Prop}$

such that for all structures $\mathcal{A}$,$\mathcal{B}$

$$\mathcal{A} \sqcup \mathcal{B} \models \beta \quad \text{iff} \quad P(x_1, \ldots, x_n, y_1, \ldots, y_n) = \text{true}$$

where $x_i = \text{true iff } \beta_i^{\mathbb{1}} \models \mathcal{A}$

$y_i = \text{true iff } \beta_i^{\mathbb{2}} \models \mathcal{B}$

$\beta ::= R(x_1, \ldots, x_n) \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$ MSO

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$E ::= x_i \mid y_i \mid E \oplus E \mid E \odot E$ Expressions

Given			$\beta \in$ MSO
there exist	$n \geq 1$	$\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{\mathbb{2}} \in \text{MSO}^n$	$P \in$ Prop

such that for all structures $\mathcal{A}$,$\mathcal{B}$

$[\![\beta]\!](\mathcal{A} \sqcup \mathcal{B}) = \text{true}$ iff $P(x_1, \ldots, x_n, y_1, \ldots, y_n) = \text{true}$

where $x_i = \text{true}$ iff $\beta_i^{\mathbb{1}} \models \mathcal{A}$

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$\beta ::= R(x_1, \ldots, x_n) \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$ MSO

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$y_i = [\![\beta_i^{2}]\!](\mathcal{B})$

$\beta ::= R(x_1, \ldots, x_n) \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$ MSO

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$E ::= x_i \mid y_i \mid E \oplus E \mid E \odot E$ Expressions

Given			$\beta \in$ MSO
there exist	$n \geq 1$	$\bar{\beta}^{1}, \bar{\beta}^{2} \in \mathrm{MSO}^n$	$E \in \mathrm{Exp}(\mathbb{B})$

such that for all structures $\mathcal{A}$,$\mathcal{B}$

$[\![\beta]\!](\mathcal{A} \sqcup \mathcal{B}) = \text{true}$ iff $P(x_1, \ldots, x_n, y_1, \ldots, y_n) = \text{true}$

where $x_i = [\![\beta_i^{1}]\!](\mathcal{A})$

$y_i = [\![\beta_i^{2}]\!](\mathcal{B})$

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Given			$\beta \in$ MSO
there exist	$n \geq 1$	$\bar{\beta}^1, \bar{\beta}^2 \in \text{MSO}^n$	$E \in \text{Exp}(\mathbb{B})$

such that for all structures $\mathcal{A}$,$\mathcal{B}$

$$[\![\beta]\!](\mathcal{A} \sqcup \mathcal{B}) = \text{true} \quad \text{iff} \quad \langle\!\langle E \rangle\!\rangle([\![\bar{\beta}^1]\!](\mathcal{A}), [\![\bar{\beta}^2]\!](\mathcal{B})) = \text{true}$$

$\beta ::= R(x_1, \ldots, x_n) \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$ MSO

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Given			$\beta \in \mathrm{MSO}$
there exist	$n \geq 1$	$\bar{\beta}^{\mathbb{1}}, \bar{\beta}^{\mathbb{2}} \in \mathrm{MSO}^n$	$E \in \mathrm{Exp}(\mathbb{B})$

such that for all structures $\mathcal{A}$,$\mathcal{B}$

$$[\![\beta]\!](\mathcal{A} \sqcup \mathcal{B}) = \langle\!\langle E \rangle\!\rangle([\![\bar{\beta}^{\mathbb{1}}]\!](\mathcal{A}), [\![\bar{\beta}^{\mathbb{2}}]\!](\mathcal{B}))$$

$\beta ::= R(x_1,\ldots,x_n) \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$ MSO

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$E ::= x_i \mid y_i \mid E \oplus E \mid E \odot E$ Expressions

Given	$(S, \oplus, \odot, \mathbb{0}, \mathbb{1})$ commutative	$\varphi \in$ wMSO(S)
there exist	$n \geq 1$	$\bar{\varphi}^1, \bar{\varphi}^2 \in$ wMSO$^n(S)$	$E \in$ Exp(S)

such that for all finite structures $\mathcal{A}$,$\mathcal{B}$

$$[\![\varphi]\!](\mathcal{A} \sqcup \mathcal{B}) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathcal{A}), [\![\bar{\varphi}^2]\!](\mathcal{B}))$$

Thm [Ravve et al. '14] [Droste, P '18]

$\beta ::= R(x_1,\dots,x_n) \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \sout{\exists X.\beta}$ FO

$P ::= x_i \mid y_i \mid P \vee P \mid P \wedge P$ Propositional formulas Prop

$\psi ::= \beta \mid s \mid \psi \oplus \psi \mid \psi \odot \psi$

$\varphi ::= \beta \mid s \mid \varphi \oplus \varphi \mid \varphi \odot \varphi \mid \bigoplus x.\varphi \mid \sout{\bigodot x.\psi \mid \bigoplus X.\varphi}$ $\mathrm{wFO}_\oplus$

$E ::= x_i \mid y_i \mid E \oplus E \mid E \odot E$ Expressions

Given	$(S,\oplus,\odot,\mathbb{0},\mathbb{1})$ commutative	$\varphi \in \mathrm{wFO}_\oplus(S)$
there exist	$n \geq 1$ $\quad \bar{\varphi}^{\mathbb{1}}, \bar{\varphi}^{2} \in \mathrm{wFO}^n_\oplus(S)$	$E \in \mathrm{Exp}(S)$

such that for all finite structures $\mathcal{A}$,$\mathcal{B}$

$$[\![\varphi]\!](\mathcal{A} \times \mathcal{B}) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathcal{A}), [\![\bar{\varphi}^{2}]\!](\mathcal{B}))$$

Thm [Droste, P '18]

$$[\![\varphi]\!](\mathcal{A} \sqcup \mathcal{B}) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathcal{A}), [\![\bar{\varphi}^2]\!](\mathcal{B}))$$

Example $\quad \mathrm{label}_a(\cdot), \mathrm{label}_b(\cdot), \mathrm{edge}(\cdot, \cdot) \quad (\mathbb{N}_0, +, \cdot, 0, 1)$

$$[\![\varphi]\!](\mathcal{A} \sqcup \mathcal{B}) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathcal{A}), [\![\bar{\varphi}^2]\!](\mathcal{B}))$$

Example $\quad \mathrm{label}_a(\cdot), \mathrm{label}_b(\cdot), \mathrm{edge}(\cdot,\cdot) \quad (\mathbb{N}_0, +, \cdot, 0, 1)$

$\varphi = |b\text{-}b\text{-edges}| \quad \cdot \quad |a\text{-vertices}|$

$$[\![\varphi]\!](\mathcal{A} \sqcup \mathcal{B}) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathcal{A}), [\![\bar{\varphi}^2]\!](\mathcal{B}))$$

Example $\text{label}_a(\cdot), \text{label}_b(\cdot), \text{edge}(\cdot, \cdot)$ $(\mathbb{N}_0, +, \cdot, 0, 1)$

$$\varphi = |b\text{-}b\text{-edges}| \cdot |a\text{-vertices}|$$

$$= \underbrace{\bigoplus x. \bigoplus y.\text{edge}(x, y) \wedge \text{label}_b(x) \wedge \text{label}_b(y)}_{\varphi_{|b-b|}} \odot \underbrace{\bigoplus z.\text{label}_a(z)}_{\varphi_{|a|}}$$

$$[\![\varphi]\!](\mathcal{A} \sqcup \mathcal{B}) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathcal{A}), [\![\bar{\varphi}^2]\!](\mathcal{B}))$$

Example

$\mathrm{label}_a(\cdot), \mathrm{label}_b(\cdot), \mathrm{edge}(\cdot, \cdot)$

$(\mathbb{N}_0, +, \cdot, 0, 1)$

$$\varphi = |b\text{-}b\text{-edges}| \cdot |a\text{-vertices}|$$

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$$\bar{\varphi}^1 = \bar{\varphi}^2 = (\varphi_{|b-b|}, \varphi_{|a|})$$

$$E = (x_1 \oplus y_1) \odot (x_2 \oplus y_2)$$

$$[\![\varphi]\!](\mathcal{A} \sqcup \mathcal{B}) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^{\mathbb{1}}]\!](\mathcal{A}), [\![\bar{\varphi}^{2}]\!](\mathcal{B}))$$

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$\bar{\varphi}^{\mathbb{1}} = \bar{\varphi}^{2} = (\varphi_{|b-b|}, \varphi_{|a|})$ $\qquad E = (x_1 \oplus y_1) \odot (x_2 \oplus y_2)$

Restrictions on $\bigodot x.\varphi$ and $\bigodot X.\varphi$ necessary $\qquad$ decomposition fails for

$$[\![\varphi]\!](\mathcal{A} \sqcup \mathcal{B}) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathcal{A}), [\![\bar{\varphi}^2]\!](\mathcal{B}))$$

Example $\quad \mathrm{label}_a(\cdot), \mathrm{label}_b(\cdot), \mathrm{edge}(\cdot,\cdot) \quad (\mathbb{N}_0, +, \cdot, 0, 1)$

$$\varphi = |b\text{-}b\text{-edges}| \cdot |a\text{-vertices}|$$
$$= \underbrace{\bigoplus x. \bigoplus y.\mathrm{edge}(x,y) \wedge \mathrm{label}_b(x) \wedge \mathrm{label}_b(y)}_{\varphi_{|b-b|}} \odot \underbrace{\bigoplus z.\mathrm{label}_a(z)}_{\varphi_{|a|}}$$

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Restrictions on $\bigodot x.\varphi$ and $\bigodot X.\varphi$ necessary $\quad$ decomposition fails for

$\bigodot x. \bigoplus y.1 \quad |A|^{|A|} \quad (\mathbb{N}_0, +, \cdot, 0, 1)$

$$[\![\varphi]\!](\mathcal{A} \sqcup \mathcal{B}) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathcal{A}), [\![\bar{\varphi}^2]\!](\mathcal{B}))$$

Example $\quad \text{label}_a(\cdot), \text{label}_b(\cdot), \text{edge}(\cdot, \cdot) \quad (\mathbb{N}_0, +, \cdot, 0, 1)$

$$\varphi = |b\text{-}b\text{-edges}| \cdot |a\text{-vertices}|$$

$$= \underbrace{\bigoplus x. \bigoplus y.\text{edge}(x, y) \wedge \text{label}_b(x) \wedge \text{label}_b(y)}_{\varphi_{|b-b|}} \odot \underbrace{\bigoplus z.\text{label}_a(z)}_{\varphi_{|a|}}$$

$\bar{\varphi}^1 = \bar{\varphi}^2 = (\varphi_{|b-b|}, \varphi_{|a|})$ $\quad E = (x_1 \oplus y_1) \odot (x_2 \oplus y_2)$

Restrictions on $\bigodot x.\varphi$ and $\bigodot X.\varphi$ necessary

		decomposition fails for
$\bigodot x. \bigoplus y.1$	$\lvert A\rvert^{\lvert A\rvert}$	$(\mathbb{N}_0, +, \cdot, 0, 1)$
$\bigodot x. \bigodot y.1$	$\lvert A\rvert^2$	$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

$$[\![\varphi]\!](\mathcal{A} \sqcup \mathcal{B}) = \langle\!\langle E \rangle\!\rangle([\![\bar{\varphi}^1]\!](\mathcal{A}), [\![\bar{\varphi}^2]\!](\mathcal{B}))$$

Example $\quad \mathrm{label}_a(\cdot), \mathrm{label}_b(\cdot), \mathrm{edge}(\cdot,\cdot) \quad (\mathbb{N}_0, +, \cdot, 0, 1)$

$$\varphi = |b\text{-}b\text{-edges}| \cdot |a\text{-vertices}|$$

$$= \underbrace{\bigoplus x. \bigoplus y. \mathrm{edge}(x,y) \wedge \mathrm{label}_b(x) \wedge \mathrm{label}_b(y)}_{\varphi_{|b-b|}} \odot \underbrace{\bigoplus z. \mathrm{label}_a(z)}_{\varphi_{|a|}}$$

$$\bar{\varphi}^1 = \bar{\varphi}^2 = (\varphi_{|b-b|}, \varphi_{|a|}) \qquad E = (x_1 \oplus y_1) \odot (x_2 \oplus y_2)$$

Restrictions on $\bigodot x.\varphi$ and $\bigodot X.\varphi$ necessary $\quad$ decomposition fails for

$\bigodot x. \bigoplus y.1$	$\|A\|^{\|A\|}$	$(\mathbb{N}_0, +, \cdot, 0, 1)$
$\bigodot x. \bigodot y.1$	$\|A\|^2$	$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$
$\bigodot X.1$	$2^{\|A\|}$	$(\mathbb{Z} \cup \{-\infty\}, \max, +, -\infty, 0)$

$$\beta ::= P_a(x) \mid x \leq y \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$$

$$\psi ::= \beta \mid s \mid \psi \oplus \psi \mid \psi \odot \psi$$

$$\varphi ::= \beta \mid s \mid \varphi \oplus \varphi \mid \varphi \odot \varphi \mid \bigoplus x.\varphi \mid \bigodot x.\psi \mid \bigoplus X.\varphi$$

restriction to first order?

restriction to only $\bigodot x.\psi$?

Fragments

$$\beta ::= P_a(x) \mid x \le y \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$$

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restriction to first order?

restriction to only $\bigodot x.\psi$?

Ambiguity of automata

$\mathrm{Run}(w) = \{\text{Runs } r \text{ on } w \text{ with weight}(r) \neq \mathbb{0}\}$

$$\beta ::= P_a(x) \mid x \leq y \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$$

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restriction to first order?

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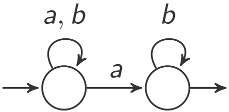
Ambiguity of automata

$\mathrm{Run}(w) = \{\text{Runs } r \text{ on } w \text{ with weight}(r) \neq \mathbb{0}\}$

unambiguous $|\mathrm{Run}(w)| \leq 1$



Fragments

$$\beta ::= P_a(x) \mid x \leq y \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$$

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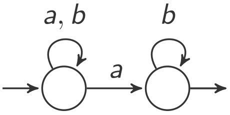
restriction to first order?

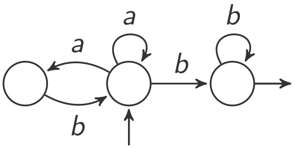
restriction to only $\bigodot x.\psi$?

Ambiguity of automata

$\mathrm{Run}(w) = \{\text{Runs } r \text{ on } w \text{ with weight}(r) \neq \mathbb{0}\}$

unambiguous	$\lvert\mathrm{Run}(w)\rvert \leq 1$
finitely ambiguous	$\lvert\mathrm{Run}(w)\rvert \leq M$





Fragments

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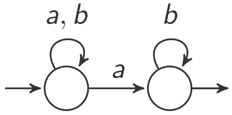
restriction to first order?

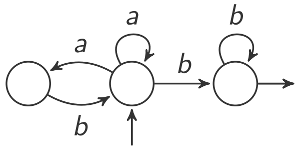
restriction to only $\bigodot x.\psi$?

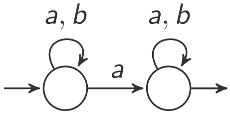
Ambiguity of automata

$\mathrm{Run}(w) = \{\text{Runs } r \text{ on } w \text{ with weight}(r) \neq \mathbb{0}\}$

unambiguous	$\lvert\mathrm{Run}(w)\rvert \leq 1$
finitely ambiguous	$\lvert\mathrm{Run}(w)\rvert \leq M$
polynomially ambiguous	$\lvert\mathrm{Run}(w)\rvert \leq P(\lvert w\rvert)$







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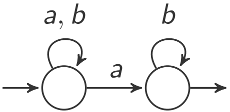
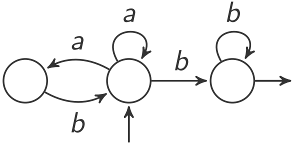
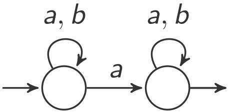
restriction to first order? restriction to only $\bigodot x.\psi$?

Ambiguity of automata $\mathrm{Run}(w) = \{\text{Runs } r \text{ on } w \text{ with weight}(r) \neq \mathbb{0}\}$

unambiguous	$\lvert\mathrm{Run}(w)\rvert \leq 1$	$\bigodot x.\psi$
finitely ambiguous	$\lvert\mathrm{Run}(w)\rvert \leq M$	
polynomially ambiguous	$\lvert\mathrm{Run}(w)\rvert \leq P(\lvert w\rvert)$	







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$$\varphi ::= \beta \mid s \mid \varphi \oplus \varphi \mid \varphi \odot \varphi \mid \bigoplus x.\varphi \mid \bigodot x.\psi \mid \bigoplus X.\varphi$$

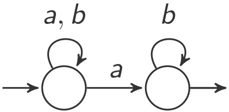
restriction to first order?

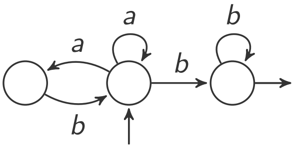
restriction to only $\bigodot x.\psi$?

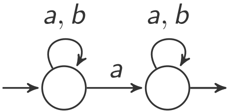
Ambiguity of automata

$\text{Run}(w) = \{\text{Runs } r \text{ on } w \text{ with weight}(r) \neq \mathbb{0}\}$

unambiguous	$\lvert\text{Run}(w)\rvert \leq 1$	$\bigodot x.\psi$
finitely ambiguous	$\lvert\text{Run}(w)\rvert \leq M$	$\bigoplus_{i=1}^{n} \bigodot x.\psi_i$
polynomially ambiguous	$\lvert\text{Run}(w)\rvert \leq P(\lvert w\rvert)$	







FRAGMENTS

$$\beta ::= P_a(x) \mid x \le y \mid x \in X \mid \neg\beta \mid \beta \vee \beta \mid \exists x.\beta \mid \exists X.\beta$$

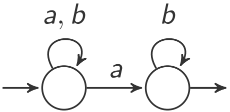
$$\psi ::= \beta \mid s \mid \psi \oplus \psi \mid \psi \odot \psi$$

$$\varphi ::= \beta \mid s \mid \varphi \oplus \varphi \mid \varphi \odot \varphi \mid \bigoplus x.\varphi \mid \bigodot x.\psi \mid \bigoplus X.\varphi$$

restriction to first order? restriction to only $\bigodot x.\psi$?

Ambiguity of automata $\mathrm{Run}(w) = \{\text{Runs } r \text{ on } w \text{ with weight}(r) \neq \mathbb{0}\}$

unambiguous	$\lvert\mathrm{Run}(w)\rvert \le 1$	$\bigodot x.\psi$
finitely ambiguous	$\lvert\mathrm{Run}(w)\rvert \le M$	$\bigoplus_{i=1}^{n} \bigodot x.\psi_i$
polynomially ambiguous	$\lvert\mathrm{Run}(w)\rvert \le P(\lvert w\rvert)$	$\bigoplus_{i=1}^{n} \bigoplus x_1 \cdots \bigoplus x_{k_n} \bigodot x.\psi_i$



[Kreutzer, Riveros '14]