Detection of Total Rotations on 2D-Vector Fields with Geometric Correlation

Roxana Bujack*, Gerik Scheuermann* and Eckhard Hitzer†

*Universität Leipzig, Institut für Informatik, Johannisgasse 26, 04103 Leipzig, Germany
†University of Fukui, Department of Applied Physics, 3-9-1 Bunkyo, Fukui 910-8507, Japan

Abstract. Correlation is a common technique for the detection of shifts. Its generalization to the multidimensional geometric correlation in Clifford algebras additionally contains information with respect to rotational misalignment. It has been proven a useful tool for the registration of vector fields that differ by an outer rotation.

In this paper we proof that applying the geometric correlation iteratively has the potential to detect the total rotational misalignment for linear two-dimensional vector fields. We further analyze its effect on general analytic vector fields and show how the rotation can be calculated from their power series expansions.

Keywords: geometric algebra, Clifford algebra, registration, total rotation, correlation, iteration.

1. INTRODUCTION

In signal processing correlation is one of the elementary techniques to measure the similarity of two input signals. It can be imagined like sliding one signal across the other and multiplying both at every shifted location. The point of registration is the very position, where the normalized cross correlation function takes its maximum, because intuitively explained there the integral is built over squared and therefore purely positive values. For a detailed proof compare [11]. Correlation is very robust and can be calculated fast using the fast Fourier transform. Therefore it is widely used for signal analysis, image registration, pattern recognition, and feature extraction [1, 13].

For quite some time the generalization of this method to multivariate data has only been parallel processing of the single channel technique. Multivectors, the elements of geometric or Clifford algebras $C\ell_{p,q}$ [4, 7] have a natural geometric interpretation. So the analysis of multidimensional signals expressed as multivector valued functions is a very reasonable approach.

Scheuermann made use of Clifford algebras for vector field analysis in [12]. Together with Ebling [5, 6] they applied geometric convolution and correlation to develop a pattern matching algorithm. They were able to accelerate it by means of a Clifford Fourier transform and the respective convolution theorem.

At about the same time Moxey, Ell, and Sangwine [9, 10] used the geometric properties of quaternions to represent color images, interpreted as vector fields. They introduced a generalized hypercomplex correlation for quaternion valued functions. Moxey et. al. state in [10], that the hypercomplex correlation of translated and outer rotated images will have its maximum peak at the position of the shift and that the correlation at this point also contains information about the outer rotation. From this they were able to approximately correct rotational distortions in color space.

In [2] we extended their work and ideas analyzing vector fields with values in the Clifford algebra $C\ell_{3,0}$ and their copies produced from outer rotations. We proved that iterative application of the rotation encoded in the cross correlation at the point of registration completely eliminates the outer misalignment of the vector fields.

In this paper we go one step further and analyze if iteration can not only lead to the detection of outer rotations but also to the detection of total rotations of vector fields.

The term rotational misalignment with respect to multivector fields is ambiguous. We distinguish three cases, visualized for a simple example in Figure 1. Let R_α be an operator, that describes a mathematically positive rotation by the angle α.

Two multivector fields $\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x}) : \mathbb{R}^m \to C\ell_{p,q}$ differ by an **inner rotation** if they suffice

$$\mathbf{A}(\mathbf{x}) = \mathbf{B}(\mathrm{R}_{-\alpha}(\mathbf{x})). \tag{1.1}$$

It can be interpreted like the starting position of every vector is rotated by α. Then the old vector is reattached at the new position, but it still points into the old direction. The inner rotation is suitable to describe the rotation of a color image. The color is represented as a vector and does not change when the picture is turned.

Another kind of misalignment we want to mention is the **outer rotation**

$$\mathbf{A}(\mathbf{x}) = \mathrm{R}_\alpha(\mathbf{B}(\mathbf{x})). \tag{1.2}$$

Here every vector on the vector field $\mathbf{A}$ is the rotated copy of every vector in the vector field $\mathbf{B}$. The vectors

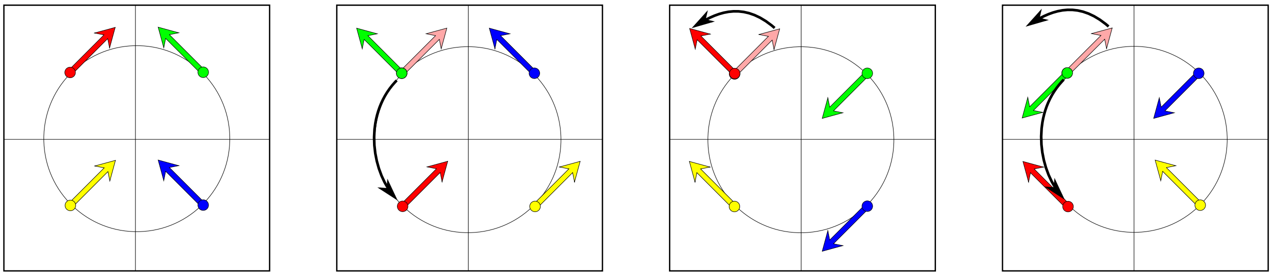

FIGURE 1. From left to right: a vector field, its copy from inner rotation, outer rotation, total rotation.

are rotated independently from their positions. This kind of rotation appears for example in color images, when the color space is turned but the picture is not moved, compare [10].

The third and in this paper most relevant kind is the **total rotation**

$$\mathbf{A}(\mathbf{x}) = \mathrm{R}_\alpha(\mathbf{B}(\mathrm{R}_{-\alpha}(\mathbf{x}))). \tag{1.3}$$

The positions and the multivectors are stiffly connected during this kind of rotation. If domain and codomain are of equal dimension it can be interpreted as a coordinate transform, as looking at the multivector field from another point of view. A total rotation is the most intuitive of the misalignments, it occurs in physical vector fields like for example fluid mechanics, and aerodynamics.

With respect to the definition of the correlation there are different formulae in current literature, [6, 10]. We prefer the following one because it satisfies a geometric generalization of the Wiener-Khinchin theorem and because it coincides with the definition of the standard cross-correlation in the special case of complex functions.

Definition 1.1. The **geometric cross correlation** of two multivector valued functions $\mathbf{A}(\mathbf{x}),\mathbf{B}(\mathbf{x}) : \mathbb{R}^m \to C\ell_{p,q}$ is a multivector valued function defined by

$$(\mathbf{A} \star \mathbf{B})(\mathbf{x}) := \int_{\mathbb{R}^m} \overline{\mathbf{A}(\mathbf{y})}\mathbf{B}(\mathbf{y}+\mathbf{x})\,\mathrm{d}^m\mathbf{y}, \tag{1.4}$$

where $\overline{\mathbf{A}(\mathbf{y})} = \sum_{k=0}^{n} (-1)^{\frac{1}{2}k(k-1)} \langle \mathbf{A}(\mathbf{y}) \rangle_k$ is the reversion.

Remark 1.2. To simplify notation we will make some conventions. Without loss of generality we assume the integrable vector fields to be normalized with respect to the L^2-norm. That way the normalized cross correlation coincides with its unnormalized counterpart. We will also only analyze the correlation at the origin. Since our vector fields are not shifted, the origin of coordinates is the place of the translational registration. If the vector fields should also differ by an inner shift, our methods can be applied analogously to this location.

2. MOTIVATION

The fundamental idea for this paper stems from the correlation of a two-dimensional vector field and its copy from outer rotation

$$\begin{aligned}(\mathrm{R}_\alpha(\mathbf{v}) \star \mathbf{v})(0) &= \int_{\mathbb{R}^2} \overline{\mathrm{R}_\alpha(\mathbf{v}(\mathbf{x}))}\mathbf{v}(\mathbf{x})\,\mathrm{d}^2\mathbf{x} \\ &= \int_{\mathbb{R}^2} \overline{e^{-\alpha\mathbf{e}_{12}}\mathbf{v}(\mathbf{x})}\mathbf{v}(\mathbf{x})\,\mathrm{d}^2\mathbf{x} \\ &= \|\mathbf{v}(\mathbf{x})\|_{L^2}^2 e^{-\alpha\mathbf{e}_{12}}.\end{aligned} \tag{2.1}$$

Since $\|\mathbf{v}(\mathbf{x})\|_{L^2}^2 \in \mathbb{R}$ the alignment can be restored by rotating back $\mathrm{R}_\alpha(\mathbf{v})$ by the angle encoded in the argument.

We want to develop this idea further to analyze total rotations. In $C\ell_{2,0}$ they take the shape

$$\mathbf{u}(\mathbf{x}) = \mathrm{R}_\alpha(\mathbf{v}(\mathrm{R}_{-\alpha}(\mathbf{x}))) = e^{-\alpha\mathbf{e}_{12}}\mathbf{v}(e^{\alpha\mathbf{e}_{12}}\mathbf{x}), \tag{2.2}$$

so it is not possible to predict the rotation that is encoded in the geometric correlation without knowing the shape of $\mathbf{v}$.

Vector fields that depend only on the magnitude of $\mathbf{x}$ are invariant with respect to inner rotations. It is easy to see that in this case the correlation takes the same shape as in (2.1) and that the misalignment can be corrected applying a rotation by the angle in the argument, too. But in general the vector fields and the rotor can not be separated from the integral of the correlation

$$\begin{aligned}(\mathbf{u} \star \mathbf{v})(0) &= \int_{\mathbb{R}^m} \overline{\mathrm{R}_\alpha(\mathbf{v}(\mathbf{x}))}\mathbf{v}(\mathbf{x})\,\mathrm{d}^m\mathbf{x} \\ &= e^{-\alpha\mathbf{e}_{12}} \int_{\mathbb{R}^m} \mathbf{v}(e^{\alpha\mathbf{e}_{12}}\mathbf{x})\mathbf{v}(\mathbf{x})\,\mathrm{d}^m\mathbf{x}.\end{aligned} \tag{2.3}$$

We dealt with a similar problem in [2] when we treated the three-dimensional outer rotation. For this case we could prove that the encoded rotation is at least a fair approximation to the one sought after and that iterative application leads to the detection of the misalignment sought after.

Trying to adapt this idea to total rotations we discovered that this result does not apply to all two-dimensional vector fields, compare the following counterexample.

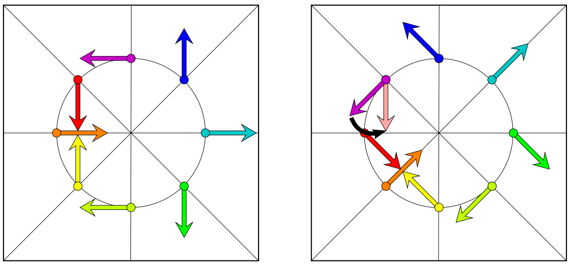

FIGURE 2. Left: vector field from the counter example. Right: mathematically positively rotated copy by $\frac{\pi}{4}$. At each position the same rotor, depicted as black arrow, contributes to the correlation.

Example. Let $\mathbf{v}: B_1(0) \to \mathbb{R}^2$ be the vector field from Figure 2 vanishing outside the unit circle take the shape

$$\mathbf{v}(r,\varphi) = \mathbf{e}_1 e^{2\varphi \mathbf{e}_{12}} \tag{2.4}$$

expressed in polar coordinates. Then its rotated copy suffices

$$\mathbf{u}(r,\varphi) = \mathbf{e}_1 e^{(2\varphi-\alpha)\mathbf{e}_{12}} \tag{2.5}$$

inside the unit circle and the correlation of the two is

$$\begin{aligned}(\mathbf{u}(\mathbf{x}) \star \mathbf{v}(\mathbf{x}))(0) &= \int_{B_1(0)} \mathbf{e}_1 e^{(2\varphi-\alpha)\mathbf{e}_{12}} \mathbf{e}_1 e^{2\varphi\mathbf{e}_{12}} r\,\mathrm{d}r\,\mathrm{d}\varphi \\ &= \int_{B_1(0)} \mathbf{e}_1\mathbf{e}_1 e^{-(2\varphi-\alpha)\mathbf{e}_{12}} e^{2\varphi\mathbf{e}_{12}} r\,\mathrm{d}r\,\mathrm{d}\varphi \\ &= \int_{B_1(0)} e^{\alpha\mathbf{e}_{12}} r\,\mathrm{d}r\,\mathrm{d}\varphi \\ &= \pi e^{\alpha\mathbf{e}_{12}}.\end{aligned} \tag{2.6}$$

If we want to correct the misalignment by rotating back with its inverse like in (2.1) we would rotate in the completely wrong direction and double the misalignment with each step, because no matter how the rotational misalignment was, we always detect its negative. So imagine starting the iterative algorithm from [2] with $\alpha = \frac{2\pi}{3}$. It would become periodic $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3} = \frac{2\pi}{3}, \ldots$ and not converge at all.

But the idea applies to all linear fields. We will show in the next sections, that iteratively rotating back with the inverse of the normalized geometric correlation will detect the correct misalignment of any two-dimensional linear vector field and its copy from total rotation.

3. LINEAR FIELDS AND ITERATIVE CORRELATION

Assume a linear vector field in two dimensions

$$\mathbf{v}(\mathbf{x}) = (a_{11}x_1 + a_{12}x_2)\mathbf{e}_1 + (a_{21}x_1 + a_{22}x_2)\mathbf{e}_2 \tag{3.1}$$

with real coefficients. Before analyzing the general linear case, let us look the examples in Figure 3, the saddles

$$\begin{aligned}\mathbf{a}(\mathbf{x}) &= x_1\mathbf{e}_1 - x_2\mathbf{e}_2, \\ \mathbf{b}(\mathbf{x}) &= x_2\mathbf{e}_1 + x_1\mathbf{e}_2,\end{aligned} \tag{3.2}$$

the source

$$\mathbf{c}(\mathbf{x}) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2, \tag{3.3}$$

and the vortex

$$\mathbf{d}(\mathbf{x}) = -x_2\mathbf{e}_1 + x_1\mathbf{e}_2. \tag{3.4}$$

Remark 3.1. Instead of using the coefficients the vector fields from Figure 3 can analogously be expressed by basic transformations

$$\begin{aligned}\mathbf{c}(\mathbf{x}) &= \mathbf{x}, \\ \mathbf{a}(\mathbf{x}) &= \mathbf{e}_1\mathbf{x}\mathbf{e}_1, \\ \mathbf{d}(\mathbf{x}) &= \mathbf{x}\mathbf{e}_{12}, \\ \mathbf{b}(\mathbf{x}) &= \mathbf{e}_1\mathbf{x}\mathbf{e}_2.\end{aligned} \tag{3.5}$$

The first three of them are the identity, a reflection at the $\mathbf{e}_1$-axis and a rotation about $\frac{\pi}{2}$. The last one $\mathbf{b}(\mathbf{x}) = \mathbf{e}_1\mathbf{x}\mathbf{e}_2 = \mathbf{e}_1\mathbf{x}\mathbf{e}_1\mathbf{e}_{12}$ as a reflection at the $\mathbf{e}_1$-axis followed by a rotation about $\frac{\pi}{2}$. From this description we immediately get

$$\begin{aligned}\mathbf{a}(\mathbf{x}) &\perp \mathbf{b}(\mathbf{x}), \\ \mathbf{c}(\mathbf{x}) &\perp \mathbf{d}(\mathbf{x}),\end{aligned} \tag{3.6}$$

and

$$\mathbf{a}(\mathbf{x})^2 = \mathbf{b}(\mathbf{x})^2 = \mathbf{c}(\mathbf{x})^2 = \mathbf{d}(\mathbf{x})^2 = \mathbf{x}^2. \tag{3.7}$$

Lemma 3.2. *Any linear vector field can be expressed as a linear combination of the four examples from the preceding section. That means for*

$$\begin{aligned}\mathbf{a}(\mathbf{x}) &= \mathbf{e}_1\mathbf{x}\mathbf{e}_1, \\ \mathbf{b}(\mathbf{x}) &= \mathbf{e}_1\mathbf{x}\mathbf{e}_2, \\ \mathbf{c}(\mathbf{x}) &= \mathbf{x}, \\ \mathbf{d}(\mathbf{x}) &= \mathbf{x}\mathbf{e}_{12}.\end{aligned} \tag{3.8}$$

there are $a,b,c,d \in \mathbb{R}$, such that

$$\mathbf{v}(x) = a\mathbf{a}(\mathbf{x}) + b\mathbf{b}(\mathbf{x}) + c\mathbf{c}(\mathbf{x}) + d\mathbf{d}(\mathbf{x}). \tag{3.9}$$

Proof. Direct calculation. □

Lemma 3.3. *Let the part in Lemma 3.2 of the two-dimensional linear vector field $\mathbf{v}(\mathbf{x})$ consisting of the two saddles be denoted by*

$$\mathbf{v}_1(\mathbf{x}) = a\mathbf{a}(\mathbf{x}) + b\mathbf{b}(\mathbf{x}) = \mathbf{e}_1\mathbf{x}(a\mathbf{e}_1 + b\mathbf{e}_2) \tag{3.10}$$

and the part consisting of the vortex and the source by

$$\mathbf{v}_2(\mathbf{x}) = c\mathbf{c}(\mathbf{x}) + d\mathbf{d}(\mathbf{x}) = \mathbf{x}\mathbf{e}_1(c\mathbf{e}_1 + d\mathbf{e}_2). \tag{3.11}$$

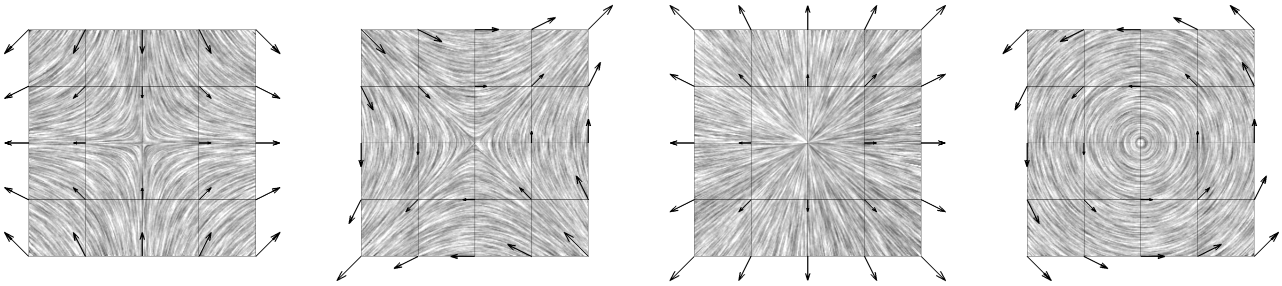

FIGURE 3. From left to right: saddles $\mathbf{a}(\mathbf{x})$, $\mathbf{b}(\mathbf{x})$, source $\mathbf{c}(\mathbf{x})$, and vortex $\mathbf{d}(\mathbf{x})$ visualized with hedgehogs and LIC [3].

Then their totally rotated copies take the shapes

$$\begin{aligned} \mathrm{R}_\alpha(\mathbf{v}_1(\mathrm{R}_{-\alpha}(\mathbf{x}))) &= e^{-2\alpha\mathbf{e}_{12}}\mathbf{v}_1(\mathbf{x}), \\ \mathrm{R}_\alpha(\mathbf{v}_2(\mathrm{R}_{-\alpha}(\mathbf{x}))) &= \mathbf{v}_2(\mathbf{x}). \end{aligned} \tag{3.12}$$

Proof. Application of the total rotation leads to

$$\begin{aligned} \mathrm{R}_\alpha(\mathbf{v}_1(\mathrm{R}_{-\alpha}(\mathbf{x}))) &= e^{-\alpha\mathbf{e}_{12}}\mathbf{e}_1 e^{\alpha\mathbf{e}_{12}}\mathbf{x}(a\mathbf{e}_1+b\mathbf{e}_2) \\ &= e^{-2\alpha\mathbf{e}_{12}}\mathbf{v}_1(\mathbf{x}), \\ \mathrm{R}_\alpha(\mathbf{v}_2(\mathrm{R}_{-\alpha}(\mathbf{x}))) &= e^{-\alpha\mathbf{e}_{12}}e^{\alpha\mathbf{e}_{12}}\mathbf{x}\mathbf{e}_1(c\mathbf{e}_1+d\mathbf{e}_2) \\ &= \mathbf{v}_2(\mathbf{x}). \end{aligned} \tag{3.13}$$

□

Lemma 3.4. *Let* $\mathbf{v}_1(\mathbf{x}) = \mathbf{e}_1\mathbf{x}(a\mathbf{e}_1+b\mathbf{e}_2), \mathbf{v}_2(\mathbf{x}) = \mathbf{x}\mathbf{e}_1(c\mathbf{e}_1+d\mathbf{e}_2)$ *be the fields from Lemma 3.3. The product of any two-dimensional linear vector field* $\mathbf{v}(\mathbf{x})$ *and its totally rotated copy* $\mathbf{u}(\mathbf{x}) = \mathrm{R}_\alpha(\mathbf{v}(\mathrm{R}_{-\alpha}(\mathbf{x})))$ *takes the shape*

$$\begin{aligned} \mathbf{u}(\mathbf{x})\mathbf{v}(\mathbf{x}) =& e^{-2\alpha\mathbf{e}_{12}}\mathbf{v}_1(\mathbf{x})^2 + e^{-2\alpha\mathbf{e}_{12}}\mathbf{v}_1(\mathbf{x})\mathbf{v}_2(\mathbf{x}) \\ &+ \mathbf{v}_2(\mathbf{x})\mathbf{v}_1(\mathbf{x}) + \mathbf{v}_2(\mathbf{x})^2 \end{aligned} \tag{3.14}$$

with

$$\begin{aligned} \mathbf{v}_1(\mathbf{x})^2 &= (a^2+b^2)(x_1^2+x_2^2), \\ \mathbf{v}_1(\mathbf{x})\mathbf{v}_2(\mathbf{x}) &= (a-b\mathbf{e}_{12})(c-d\mathbf{e}_{12})(x_1^2-x_2^2+2x_1x_2\mathbf{e}_{12}), \\ \mathbf{v}_2(\mathbf{x})\mathbf{v}_1(\mathbf{x}) &= (a+b\mathbf{e}_{12})(c+d\mathbf{e}_{12})(x_1^2-x_2^2-2x_1x_2\mathbf{e}_{12}), \\ \mathbf{v}_2(\mathbf{x})^2 &= (c^2+d^2)(x_1^2+x_2^2). \end{aligned} \tag{3.15}$$

Proof. We know from Lemmata 3.2 and 3.3 that the vector field can be split into $\mathbf{v}(x) = \mathbf{v}_1(\mathbf{x}) + \mathbf{v}_2(\mathbf{x})$. Because of the linearity of the rotation and Lemma 3.3 we get

$$\mathrm{R}_\alpha(\mathbf{v}(\mathrm{R}_{-\alpha}(\mathbf{x}))) = e^{-2\alpha\mathbf{e}_{12}}\mathbf{v}_1(\mathbf{x}) + \mathbf{v}_2(\mathbf{x}) \tag{3.16}$$

and therefore the product suffices (3.14). The assertions about the exact shape of the summands follow from straight calculation. □

The argument φ of the geometric product (3.14) is not generally a good approximation to $-\alpha$. But we will show that it is always in $[0,-2\alpha]$ if we take the integral of the product over an area A symmetric with respect to both coordinate axes, like a square or a circle. This integral is equivalent to the correlation at the origin, if we assume the vector fields to vanish outside this area.

Theorem 3.5. *Let the two-dimensional vector field* $\mathbf{v}(\mathbf{x})$ *be linear within and zero outside of an area A symmetric with respect to both coordinate axes. The correlation at the origin with its totally rotated copy* $\mathbf{u}(\mathbf{x}) = \mathrm{R}_\alpha(\mathbf{v}(\mathrm{R}_{-\alpha}(\mathbf{x})))$ *satisfies*

$$(\mathbf{u}\star\mathbf{v})(0) = e^{-2\alpha\mathbf{e}_{12}}||\mathbf{v}_1(\mathbf{x})||^2_{L^2(A)} + ||\mathbf{v}_2(\mathbf{x})||^2_{L^2(A)} \tag{3.17}$$

with $\mathbf{v}_1(\mathbf{x}) = (a-b\mathbf{e}_{12})(-\mathbf{e}_2\mathbf{x}\mathbf{e}_2), \mathbf{v}_2(\mathbf{x}) = (c+d\mathbf{e}_{12})\mathbf{x}$ *from Lemma 3.3.*

Proof. We already know from Lemma 3.4 that the product of the vector field and its rotated copy takes the form (3.14). Taking into account (3.15) and the fact, that the integral over the symmetric domain A over $x_1^2-x_2^2$ is zero as well as the integral over x_1x_2, we get

$$\begin{aligned} \int_A \mathbf{v}_1(\mathbf{x})\mathbf{v}_2(\mathbf{x})\,\mathrm{d}^2x &= 0, \\ \int_A \mathbf{v}_2(\mathbf{x})\mathbf{v}_1(\mathbf{x})\,\mathrm{d}^2 &= 0. \end{aligned} \tag{3.18}$$

That is why the integral over the product reduces to

$$\begin{aligned} \int_A \mathbf{u}(\mathbf{x})\mathbf{v}(\mathbf{x})\,\mathrm{d}^2x = \int_A & e^{-2\alpha\mathbf{e}_{12}}\mathbf{v}_1(\mathbf{x})^2 + \mathbf{v}_2(\mathbf{x})\mathbf{v}_1(\mathbf{x}) \\ &+ e^{-2\alpha\mathbf{e}_{12}}\mathbf{v}_1(\mathbf{x})\mathbf{v}_2(\mathbf{x}) + \mathbf{v}_2(\mathbf{x})^2\,\mathrm{d}^2\mathbf{x} \\ =& e^{-2\alpha\mathbf{e}_{12}}||\mathbf{v}_1(\mathbf{x})||^2_{L^2(A)} + ||\mathbf{v}_2(\mathbf{x})||^2_{L^2(A)}. \end{aligned} \tag{3.19}$$

□

Remark 3.6. Please note that an integral over an unsymmetric area does in general not lead to a result without the mixed terms $\mathbf{v}_1(\mathbf{x})\mathbf{v}_2(\mathbf{x})$.

Lemma 3.7. *Let the two-dimensional vector field* $\mathbf{v}(\mathbf{x})$ *be linear within and zero outside of an area A symmetric with respect to both coordinate axes. The angle* φ *which is the argument of the correlation at the origin with its totally rotated copy* $\mathbf{u}(\mathbf{x}) = \mathrm{R}_\alpha(\mathbf{v}(\mathrm{R}_{-\alpha}(\mathbf{x})))$ *satisfies*

$$\begin{aligned} &0 \geq \varphi \geq -2\alpha, \textit{for } \alpha \geq 0, \\ &0 \leq \varphi \leq -2\alpha, \textit{else}. \end{aligned} \tag{3.20}$$

The proof of Lemma 3.7 is very technical. Figure 4 provides a more fundamental insight of its assertion by exploiting the homomorphism of the rotors in $C\ell_{2,0}$ and the complex numbers.

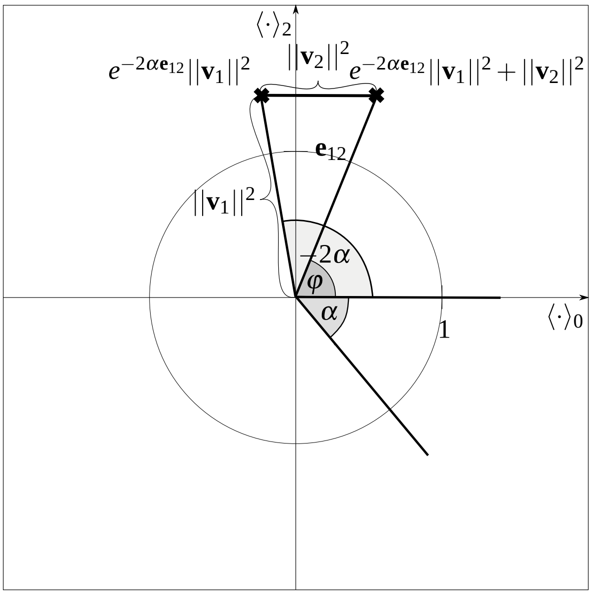

FIGURE 4. Lemma 3.7 visualized like the complex plane. Vertical axis: bivector part, horizontal axis: scalarpart.

Proof. The argument satisfies

$$\begin{aligned} \varphi &= \arg\Big(\int_{[-l,l]^2} \mathrm{R}_\alpha(\mathbf{v}(\mathrm{R}_{-\alpha}(\mathbf{x})))\mathbf{v}(\mathbf{x})\,\mathrm{d}^2x\Big) \\ &= \arg(e^{-2\alpha\mathbf{e}_{12}}||\mathbf{v}_1(\mathbf{x})||^2 + ||\mathbf{v}_2(\mathbf{x})||^2) \\ &= \operatorname{atan2}(-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2, \\ &\quad \cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 + ||\mathbf{v}_2(\mathbf{x})||^2) \end{aligned} \tag{3.21}$$

For $||\mathbf{v}_1(\mathbf{x})||^2 = 0$ and $||\mathbf{v}_2(\mathbf{x})||^2 = 0$ the statement is trivially true, because then $\varphi = 0$ or $\varphi = -2\alpha$. So let $||\mathbf{v}_1(\mathbf{x})||^2, ||\mathbf{v}_2(\mathbf{x})||^2 > 0$. Now we have to make a case differentiation.

1. The assumptions $\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 + ||\mathbf{v}_2(\mathbf{x})||^2 > 0$ and $-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 > 0$ lead to

$$\varphi = \arctan\frac{-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2}{\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 + ||\mathbf{v}_2(\mathbf{x})||^2} \tag{3.22}$$

so φ is positive. If we leave out $||\mathbf{v}_2(\mathbf{x})||^2$ the denominator gets smaller. If the denominator $\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 > 0$ remains positive the positive fraction gets larger and we have

$$\varphi \leq \arctan\frac{-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2}{\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2} = -2\alpha, \tag{3.23}$$

with positive -2α and therefore negative α. If the denominator $\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 \leq 0$ becomes negative we have $-2\alpha \in [\frac{\pi}{2}, \pi]$, because of $-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 > 0$, so $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \leq -2\alpha$.

2. The assumptions $\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 + ||\mathbf{v}_2(\mathbf{x})||^2 > 0$ and $-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 < 0$ lead to

$$\varphi = \arctan\frac{-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2}{\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 + ||\mathbf{v}_2(\mathbf{x})||^2} \tag{3.24}$$

so φ is negative. If we leave out $||\mathbf{v}_2(\mathbf{x})||^2$ the denominator gets smaller. If the denominator $\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 > 0$ remains positive the negative fraction gets smaller and we have

$$\varphi \geq \arctan\frac{-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2}{\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2} = -2\alpha, \tag{3.25}$$

with negative -2α and therefore positive α. If the denominator $\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 \leq 0$ becomes negative we have $-2\alpha \in [-\pi, -\frac{\pi}{2}]$, because of $-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 < 0$, so $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \geq -2\alpha$.

3. The assumptions $\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 + ||\mathbf{v}_2(\mathbf{x})||^2 < 0$ and $-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 > 0$ lead to

$$\varphi = \arctan\frac{-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2}{\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 + ||\mathbf{v}_2(\mathbf{x})||^2} + \pi \tag{3.26}$$

so φ is positive. If we leave out $||\mathbf{v}_2(\mathbf{x})||^2$ the magnitude of the denominator gets larger so the magnitude of the fraction gets smaller. Since the fraction is negative and the arctangent is monotonic increasing a lower magnitude increases the whole right side and we have

$$\varphi \leq \arctan\frac{-\sin(2\alpha)}{\cos(2\alpha)} + \pi = -2\alpha. \tag{3.27}$$

Because the numerator is positive and the denominator is negative this equals -2α, which is positive and therefore α is negative.

4. The assumptions $\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 + ||\mathbf{v}_2(\mathbf{x})||^2 < 0$ and $-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 < 0$ lead to

$$\varphi = \arctan\frac{-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2}{\cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 + ||\mathbf{v}_2(\mathbf{x})||^2} - \pi \tag{3.28}$$

so φ is negative. If we leave out $||\mathbf{v}_2(\mathbf{x})||^2$ the magnitude of the denominator gets larger so the magnitude of the fraction decreases. It is positive so the fraction gets smaller, so does the arctangent and the whole right side and we have

$$\varphi \geq \arctan \frac{-\sin(2\alpha)}{\cos(2\alpha)} - \pi = -2\alpha. \tag{3.29}$$

Because the numerator and the denominator are negative this equals -2α, which is negative and therefore α is positive.

Since we covered all possible configurations, we see that α and φ always have different signs. The right estimation for positive α is a result of the even cases and for negative α of the odd ones. □

Theorem 3.8. *Let the two-dimensional vector field* $\mathbf{v}(\mathbf{x})$ *be linear within and zero outside of an area A symmetric with respect to both coordinate axes and* $\varphi : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ *be the function defined by the rule*

$$\varphi(\alpha) = \arg((\mathrm{R}_\alpha(\mathbf{v}(\mathrm{R}_{-\alpha})) \star \mathbf{v})(0). \tag{3.30}$$

Then the series $\tilde{\alpha}_0 = 0, \tilde{\alpha}_{n+1} = \tilde{\alpha}_n - \varphi(\alpha - \tilde{\alpha}_n)$ *converges to* α *for all* $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$*, if* $||\mathbf{v}_1(\mathbf{x})||^2 \neq 0 \neq ||\mathbf{v}_2(\mathbf{x})||^2$.

Proof. To prove the theorem we show that the series $\alpha_n = \alpha - \tilde{\alpha}_n$ of the remaining misalignment converges to zero. It suffices

$$\begin{aligned} \alpha_0 =& \alpha - \tilde{\alpha}_0 = \alpha, \\ \alpha_{n+1} =& \alpha - \tilde{\alpha}_{n+1} = \alpha - \tilde{\alpha}_n + \varphi(\alpha - \tilde{\alpha}_n) = \alpha_n + \varphi(\alpha_n). \end{aligned} \tag{3.31}$$

Lemma 3.7 shows that the series α_n decreases with respect to its magnitude, because for $\alpha_n \in (-\frac{\pi}{2}, 0)$ we have $0 \leq \varphi(\alpha_n) \leq -2\alpha_n$ and therefore

$$\begin{aligned} \alpha_n = \alpha_n + 0 \leq \alpha_n + \varphi(\alpha_n) = \alpha_{n+1}, \\ \alpha_{n+1} = \alpha_n + \varphi(\alpha_n) \leq \alpha_n - 2\alpha_n = -\alpha_n \end{aligned} \tag{3.32}$$

and for $\alpha_n \in (0, \frac{\pi}{2})$ we have $0 \geq \varphi(\alpha_n) \geq -2\alpha_n$ and therefore

$$\begin{aligned} \alpha_n = \alpha_n + 0 \geq \alpha_n + \varphi(\alpha_n) = \alpha_{n+1}, \\ \alpha_{n+1} = \alpha_n + \varphi(\alpha_n) \geq \alpha_n - 2\alpha_n = -\alpha_n. \end{aligned} \tag{3.33}$$

Since the series of magnitudes is monotonically decreasing and bounded from below by zero it is convergent.

Let the limit of the sequence of magnitudes be $a = \lim_{n\to\infty} |\alpha_n|$ then using the definition of the series and applying the limit leads to

$$\lim_{n\to\infty}(|\alpha_{n+1}|) = \lim_{n\to\infty}(|\alpha_n + \varphi(\alpha_n)|). \tag{3.34}$$

The modulus function and $\varphi(\alpha_n)$ are continuous in $\alpha_n \in (-\frac{\pi}{2}, \frac{\pi}{2})$. That allows us to swap the limit and the functions and write

$$\begin{aligned} a =& |\lim_{n\to\infty}(\alpha_n) + \lim_{n\to\infty}(\varphi(\alpha_n))| \\ =& |a + \varphi(a)|. \end{aligned} \tag{3.35}$$

We apply a case differentiation to the previous equation.

1. For $a + \varphi(a) \geq 0$ it is equivalent to

$$a = a + \varphi(a) \Leftrightarrow \varphi(a) = 0. \tag{3.36}$$

   Since

$$\begin{aligned} \varphi(\alpha) =& \operatorname{atan2}(-\sin(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2, \\ & \cos(2\alpha)||\mathbf{v}_1(\mathbf{x})||^2 + ||\mathbf{v}_2(\mathbf{x})||^2) \end{aligned} \tag{3.37}$$

   the claim $\varphi(a) = 0$ is true for $\cos(2a)||\mathbf{v}_1(\mathbf{x})||^2 + ||\mathbf{v}_2(\mathbf{x})||^2 > 0, -\sin(2a)||\mathbf{v}_1(\mathbf{x})||^2 = 0$ which is fulfilled either for $||\mathbf{v}_1(\mathbf{x})||^2 = 0$ and arbitrary a or for $||\mathbf{v}_1(\mathbf{x})||^2 > 0$ and $a = 0$.
2. $a + \varphi(a) < 0$ leads to

$$a = -a - \varphi(a) \Leftrightarrow \varphi(a) = -2a, \tag{3.38}$$

   which is only fulfilled for $||\mathbf{v}_2(\mathbf{x})||^2 = 0$ and arbitrary a.

Combination of the two cases leads to the proposition $a = 0$ if $||\mathbf{v}_1(\mathbf{x})||^2 \neq 0 \neq ||\mathbf{v}_2(\mathbf{x})||^2$. Since the sequence of the magnitudes converges to zero the sequence itself converges to zero as well. □

From Theorem 3.8 we can construct Algorithm 1, which also converges for $||\mathbf{v}_1(\mathbf{x})||^2 = 0$, $||\mathbf{v}_2(\mathbf{x})||^2 = 0$, and any rotational misalignment α. The claim $||\mathbf{v}_1(\mathbf{x})||^2 = 0$ means $\mathbf{v}_1(\mathbf{x}) = 0$ almost everywhere. For a linear vector field this is equivalent to $\mathbf{v}_1(\mathbf{x}) = 0$, analogously $||\mathbf{v}_2(\mathbf{x})||^2 = 0 \Leftrightarrow \mathbf{v}_2(\mathbf{x}) = 0$. In the case $\mathbf{v}_1(\mathbf{x}) = 0$ Lemma 3.3 shows that $\forall\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}) : \varphi(\alpha) = 0$. An iterative algorithm would stop after one step and return the correct result, because these vector fields are rotational invariant anyway. In the case $\mathbf{v}_2(\mathbf{x}) = 0$ Lemma 3.3 shows that $\forall\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}) : \varphi(\alpha) = -2\alpha$. The algorithm would alternate between -2α and zero. That means if the algorithm takes the value zero in the α variable after its first iteration the underlying vector field must be a saddle $\mathbf{v}(\mathbf{x}) = \mathbf{v}_1(\mathbf{x})$ and the correct misalignment is half the calculated φ. This exception is handled in Line 11 in Algorithm 1. Because of the symmetry of linear vector fields the misalignment can always be described by an angle $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. In the case of $\alpha = \pm\frac{\pi}{2}$ the correlation will be real valued, compare Theorem 3.5. This case can only appear in the first step of the algorithm. It would return the angle zero like in the case where in deed no rotation is necessary. Therefore we need to include another exception handling. We suggest to apply a total rotation by $\frac{\pi}{4}$ to the pattern, if the first step returns $\alpha = 0$, compare Line 7 in Algorithm 1. The disadvantage of this treatment is that it might disturb the alignment in the nice case, when vector field and pattern incidentally match at the beginning, but will guarantee the convergence. The last exception to be treated appears when both $\alpha \in \{-\frac{\pi}{2}, 0, \frac{\pi}{2}\}$ and $\mathbf{v}(\mathbf{x}) = \mathbf{v}_1(\mathbf{x})$. In this case α gets the value $\frac{\pi}{4}$ from the first exception handling and will alternate between $\pm\frac{\pi}{4}$ for the rest of the algorithm. We fixed this problem in Line 14 in Algorithm 1.

Algorithm 1 Detection of total misalignment of vector fields

Input: vector field: $\mathbf{v}(\mathbf{x})$, rotated pattern: $\mathbf{u}(\mathbf{x})$, desired accuracy: $\varepsilon > 0$,

1: $\varphi = \pi, \alpha = 0, iter = 0, exception = false$,
2: **while** $\varphi > \varepsilon$ **do**
3:   $iter{+}{+}$,
4:   $Cor = (\mathbf{u}(\mathbf{x}) \star \mathbf{v}(\mathbf{x}))(0)$,
5:   $\varphi = \arg(Cor)$,
6:   $\alpha = \alpha - \varphi$,
7:   **if** $iter = 1$ and $\alpha = 0$ **then**
8:     $\alpha = -\pi/4, \varphi = \pi/4$,
9:     $exception = true$,
10:   **end if**
11:   **if** $iter = 2$ and not $exception$ and $\alpha = 0$ **then**
12:     $\alpha = -\varphi/2, \varphi = \varphi/2$,
13:   **end if**
14:   **if** $iter = 2$ and $exception$ and $\varphi = -\pi/2$ **then**
15:     $\alpha = -\pi/2, \varphi = \pi/4$
16:   **end if**
17:   $\mathbf{u}(\mathbf{x}) = e^{-\varphi \mathbf{e}_{12}} \mathbf{u}(e^{\varphi \mathbf{e}_{12}} \mathbf{x})$,
18: **end while**

Output: misalignment: α, corrected pattern: $\mathbf{u}(\mathbf{x})$, iterations needed: $iter$.

We practically tested Algorithm 1 applying it to continuous, linear vector fields $\mathbb{R}^2 \to C\ell_{2,0}$, that vanish outside the unit square. The experiments showed that Algorithm 1 converges in all cases, just as the theory suggested.

Remark 3.9. Theorem 3.8 is only theoretically interesting. The calculation of the misalignment of linear fields $\mathbf{u}(\mathbf{x}) = \mathrm{R}_\alpha(\mathbf{v}(\mathrm{R}_{-\alpha}(\mathbf{x})))$ is far easier. Because of Lemma 3.3 they suffice

$$\begin{aligned} \mathrm{R}_{\frac{\pi}{2}}(\mathbf{v}_1(\mathrm{R}_{-\frac{\pi}{2}}(\mathbf{x}))) &= -\mathbf{v}_1(\mathbf{x}), \\ \mathrm{R}_{\frac{\pi}{2}}(\mathbf{v}_2(\mathrm{R}_{-\frac{\pi}{2}}(\mathbf{x}))) &= \mathbf{v}_2(\mathbf{x}), \end{aligned} \tag{3.39}$$

which leads to

$$\begin{aligned} \mathbf{v}_1(\mathbf{x}) &= \frac{1}{2}\big(\mathbf{v}(\mathbf{x}) - \mathrm{R}_{\frac{\pi}{2}}(\mathbf{v}(\mathrm{R}_{-\frac{\pi}{2}}(\mathbf{x})))\big), \\ \mathbf{v}_2(\mathbf{x}) &= \frac{1}{2}\big(\mathbf{v}(\mathbf{x}) + \mathrm{R}_{\frac{\pi}{2}}(\mathbf{v}(\mathrm{R}_{-\frac{\pi}{2}}(\mathbf{x})))\big). \end{aligned} \tag{3.40}$$

Once we know the shape of $\mathbf{v}_1, \mathbf{u}_1$ the angle α can be detected easily using Lemma 3.3 from

$$\alpha = -\frac{1}{2}\arg((\mathbf{u}_1 \star \mathbf{v}_1)(0)). \tag{3.41}$$

4. GEOMETRIC VECTOR FIELD BASIS

In order to treat the total rotation of more general vector fields, we consider an idea of Liu and Ribeiro [8]. They made use of the isomorphism of 2D vector fields and complex functions and the expansion of holomorphic functions into a power series

$$\mathbf{v}(\mathbf{x}) \sim f(z) = \sum_{k=0}^{\infty} f_k z^k \tag{4.1}$$

with Taylor coefficients $f_k, \in \mathbb{C}$

$$f_k = \frac{f^{(k)}(0)}{k!}. \tag{4.2}$$

Because of the orthogonality of z^k the coefficients f_k can alternatively be calculated from correlation

$$f_k = \frac{(f(z) \star z^k)(0)}{(z^k \star z^k)(0)}. \tag{4.3}$$

Under total rotation $g(\mathbf{x}) = \mathrm{R}_\alpha(f(\mathrm{R}_{-\alpha}(\mathbf{x})))$ they behave very nicely satisfying

$$g_k = e^{\alpha \mathbf{e}_{12}(1-k)} f_k. \tag{4.4}$$

A great disadvantage of the previous description of the vector fields as holomorphic functions is that the set of holomorphic functions is very limited. Even simple vector fields like the saddle $\mathbf{a}(\mathbf{x}) = x_1\mathbf{e}_1 - x_2\mathbf{e}_2$ are not holomorphic. To solve this problem Scheuermann, [12] suggests to express the vector fields by means of two not independent complex variables $z, \bar{z}$. With this construction far more fields, namely all analytic fields in x_1, x_2

$$\mathbf{v}(x_1, x_2) = \sum_{k,l=0}^{\infty} \mathbf{v}_{k,l} x_1^k x_2^l. \tag{4.5}$$

with $\mathbf{v}_{k,l} = v_{k,l_1}\mathbf{e}_1 + v_{k,l_2}\mathbf{e}_2 \in C\ell_{2,0}, v_{k,l_1}, v_{k,l_2} \in \mathbb{R}$ can be expressed.

We adapt this idea but make use of the richness of Clifford algebras. They allow us the expansion of any 2D analytic field with respect to a geometric basis $\mathbf{e}_1\mathbf{x}, \mathbf{x}\mathbf{e}_1$ using

$$\begin{aligned} x_1 &= \mathbf{x} \cdot \mathbf{e}_1 = \frac{1}{2}(\mathbf{x}\mathbf{e}_1 + \mathbf{e}_1\mathbf{x}), \\ x_2 &= \mathbf{x} \cdot \mathbf{e}_2 = \frac{1}{2}(\mathbf{x}\mathbf{e}_2 + \mathbf{e}_2\mathbf{x}) = \frac{\mathbf{e}_{12}}{2}(\mathbf{x}\mathbf{e}_1 - \mathbf{e}_1\mathbf{x}). \end{aligned} \tag{4.6}$$

Theorem 4.1. *Every 2D analytic field in x_1, x_2*

$$\mathbf{v}(\mathbf{x}) = \sum_{k',l'=0}^{\infty} \mathbf{v}'_{k',l'} x_1^{k'} x_2^{l'} \tag{4.7}$$

can be expanded into a power series of $\mathbf{e}_1\mathbf{x}, \mathbf{x}\mathbf{e}_1$

$$\mathbf{v}(\mathbf{x}) = \sum_{k,l=0}^{\infty} (\mathbf{e}_1\mathbf{x})^k (\mathbf{x}\mathbf{e}_1)^l \mathbf{v}_{kl}. \tag{4.8}$$

and the coefficients are related by

$$\mathbf{v}_{k,l} = \sum_{m=0}^{k}\sum_{n=0}^{l} (\frac{1}{2})^{k+l} \binom{l-n+m}{m} \binom{k-m+n}{n} (-1)^{k-m} \mathbf{e}_{12}^{k-m+n} \mathbf{v}'_{l-n+m,k-m+n}. \tag{4.9}$$

Proof. We will make use of the commutation properties

$$\begin{aligned}(\mathbf{x}\mathbf{e}_1)(\mathbf{e}_1\mathbf{x}) &= (\mathbf{e}_1\mathbf{x})(\mathbf{x}\mathbf{e}_1),\\ \mathbf{e}_{12}(\mathbf{x}\mathbf{e}_1) &= (\mathbf{x}\mathbf{e}_1)\mathbf{e}_{12},\\ \mathbf{e}_{12}(\mathbf{e}_1\mathbf{x}) &= (\mathbf{e}_1\mathbf{x})\mathbf{e}_{12},\\ \mathbf{v}_{k,l}(\mathbf{x}\mathbf{e}_1) &= (\mathbf{e}_1\mathbf{x})\mathbf{v}_{k,l},\\ \mathbf{v}_{k,l}(\mathbf{e}_1\mathbf{x}) &= (\mathbf{x}\mathbf{e}_1)\mathbf{v}_{k,l},\end{aligned} \tag{4.10}$$

that can be easily checked, further for changing the limits of addition

$$\forall k < m : \binom{k}{m} = 0, \qquad \forall l < n : \binom{l}{n} = 0, \tag{4.11}$$

and finally

$$\binom{l-n+m}{m} \neq 0 \Leftrightarrow l-n+m \geq m \Leftrightarrow l \geq n \geq 0,$$
$$\binom{k-m+n}{n} \neq 0 \Leftrightarrow k-m+n \geq n \Leftrightarrow k \geq m \geq 0. \tag{4.12}$$

Then we get

$$\begin{aligned}\mathbf{v}(\mathbf{x}) &= \sum_{k',l'=0}^{\infty} x_1^{k'} x_2^{l'} \mathbf{v}'_{k',l'}\\ &\overset{(4.6)}{=} \sum_{k',l'=0}^{\infty} (\frac{1}{2}(\mathbf{x}\mathbf{e}_1 + \mathbf{e}_1\mathbf{x}))^{k'} (\frac{\mathbf{e}_{12}}{2}(\mathbf{x}\mathbf{e}_1 - \mathbf{e}_1\mathbf{x}))^{l'} \mathbf{v}'_{k',l'}\\ &\overset{(4.10)}{=} \sum_{k',l''=0}^{\infty} (\frac{1}{2})^{k'+l''} \sum_{m=0}^{k'} \binom{k'}{m} (\mathbf{x}\mathbf{e}_1)^{k'-m} (\mathbf{e}_1\mathbf{x})^m \\ &\quad \sum_{n=0}^{l'} \binom{l'}{n} (\mathbf{x}\mathbf{e}_1)^n (-\mathbf{e}_1\mathbf{x})^{l'-n} \mathbf{e}_{12}^{l'} \mathbf{v}'_{k',l'}\end{aligned} \tag{4.13}$$

$$\begin{aligned} &= \sum_{k',l'=0}^{\infty} \sum_{m=0}^{k'} \sum_{n=0}^{l'} (\frac{1}{2})^{k'+l'} \binom{k'}{m} \binom{l'}{n} \\ &\quad (\mathbf{x}\mathbf{e}_1)^{k'-m+n} (\mathbf{e}_1\mathbf{x})^{m+l'-n} (-1)^{l'-n} \mathbf{e}_{12}^{l'} \mathbf{v}'_{k',l'}\\ &\overset{(4.11)}{=} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k'=0}^{\infty}\sum_{l'=0}^{\infty} (\frac{1}{2})^{k'+l'} \binom{k'}{m} \binom{l'}{n} (-1)^{l'-n} \mathbf{e}_{12}^{l'}\\ &\quad (\mathbf{x}\mathbf{e}_1)^{k'-m+n} (\mathbf{e}_1\mathbf{x})^{m+l'-n} \mathbf{v}'_{k',l'}\\ &\overset{\substack{k=m+l'-n,\\ l=k'-m+n}}{=} \sum_{m,n=0}^{\infty} \sum_{k=m-n}^{\infty} \sum_{l=n-m}^{\infty} (\frac{1}{2})^{k+l} \binom{l-n+m}{m} \binom{k-m+n}{n}\\ &\quad (-1)^{k-m} \mathbf{e}_{12}^{k-m+n} (\mathbf{x}\mathbf{e}_1)^l (\mathbf{e}_1\mathbf{x})^k \mathbf{v}'_{l-n+m,k-m+n}\\ &\overset{(4.12)}{=} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m,n=0}^{\infty} (\frac{1}{2})^{k+l} \binom{l-n+m}{m} \binom{k-m+n}{n}\\ &\quad (-1)^{k-m} \mathbf{e}_{12}^{k-m+n} (\mathbf{x}\mathbf{e}_1)^l (\mathbf{e}_1\mathbf{x})^k \mathbf{v}'_{l-n+m,k-m+n}\\ &= \sum_{k,l=0}^{\infty} (\mathbf{e}_1\mathbf{x})^k (\mathbf{x}\mathbf{e}_1)^l \mathbf{v}_{k,l}\end{aligned} \tag{4.14}$$

with

$$\begin{aligned}\mathbf{v}_{k,l} &= \sum_{m,n=0}^{\infty} (\frac{1}{2})^{k+l} \binom{l-n+m}{m} \binom{k-m+n}{n}\\ &\quad (-1)^{k-m} \mathbf{e}_{12}^{k-m+n} \mathbf{v}'_{l-n+m,k-m+n}\\ &\overset{(4.12)}{=} \sum_{m=0}^{k}\sum_{n=0}^{l} (\frac{1}{2})^{k+l} \binom{l-n+m}{m} \binom{k-m+n}{n}\\ &\quad (-1)^{k-m} \mathbf{e}_{12}^{k-m+n} \mathbf{v}'_{l-n+m,k-m+n}.\end{aligned} \tag{4.15}$$

□

Theorem 4.2. *Let $\mathbf{u}(\mathbf{x}) = R_\alpha(\mathbf{v}(R_\alpha^{-1}(\mathbf{x})))$ be the totally rotated copy of the analytic 2D vector field $\mathbf{v}$ and $\mathbf{u}_{k,l} \neq 0 \neq \mathbf{v}_{k,l}$ their coefficients with respect to the expansion in Theorem 4.1, then their rotational misalignment α can be calculated $\forall l-k \neq 1$ from*

$$\alpha = \frac{\arg(\mathbf{u}_{k,l}\mathbf{v}_{k,l})}{l-k-1}. \tag{4.16}$$

Proof. Under total rotation

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= R_\alpha(\mathbf{v}(R_\alpha^{-1}(\mathbf{x})))\\ &= e^{-\alpha\mathbf{e}_{12}} \sum_{k,l=0}^{\infty} (\mathbf{e}_1 e^{\alpha\mathbf{e}_{12}}\mathbf{x})^k (e^{\alpha\mathbf{e}_{12}}\mathbf{x}\mathbf{e}_1)^l \mathbf{v}_{kl}\\ &= e^{-\alpha\mathbf{e}_{12}} \sum_{k,l=0}^{\infty} (\mathbf{e}_1\mathbf{x}e^{-\alpha\mathbf{e}_{12}})^k (e^{\alpha\mathbf{e}_{12}}\mathbf{x}\mathbf{e}_1)^l \mathbf{v}_{kl}\\ &\overset{4.10}{=} \sum_{k,l=0}^{\infty} (\mathbf{e}_1\mathbf{x})^k (e^{-\alpha\mathbf{e}_{12}})^k (e^{\alpha\mathbf{e}_{12}})^l (\mathbf{x}\mathbf{e}_1)^l \mathbf{v}_{kl}\\ &\overset{4.10}{=} \sum_{k,l=0}^{\infty} (\mathbf{e}_1\mathbf{x})^k (\mathbf{x}\mathbf{e}_1)^l e^{\alpha\mathbf{e}_{12}(l-k-1)} \mathbf{v}_{kl}\end{aligned} \tag{4.17}$$

the coefficients interact by

$$\mathbf{u}_{k,l} = e^{\alpha \mathbf{e}_{12}(l-k-1)} \mathbf{v}_{k,l} \tag{4.18}$$

so $\forall l-k \neq 1$ the misalignment can be calculated from

$$\alpha = \frac{\arg(\mathbf{u}_{k,l}\mathbf{v}_{k,l})}{l-k-1}. \tag{4.19}$$

□

5. CORRELATION WITH THE GEOMETRIC BASIS

In contrast to the complex monomias in z the geometric basis functions $(\mathbf{e}_1\mathbf{x})^k(\mathbf{x}\mathbf{e}_1)^l$ are not orthogonal. So the coefficients do in general not coincide with the correlation of a vector field with the basis functions

$$\frac{(\mathbf{v}(\mathbf{x}) \star (\mathbf{e}_1\mathbf{x})^k(\mathbf{x}\mathbf{e}_1)^l)(0)}{((\mathbf{e}_1\mathbf{x})^k(\mathbf{x}\mathbf{e}_1)^l \star (\mathbf{e}_1\mathbf{x})^k(\mathbf{x}\mathbf{e}_1)^l)(0)} \overset{\text{gen.}}{\neq} \mathbf{v}_{k,l}. \tag{5.1}$$

Still we can work with the correlation in the same way.

Theorem 5.1. *Let $\mathbf{v}(\mathbf{x})$ be an analytic vector field, $\mathbf{u}(\mathbf{x}) = R_\alpha(\mathbf{v}(R_\alpha^{-1}(\mathbf{x})))$ its totally rotated copy and for $k,l \in \mathbb{N}, l-k \neq 1$ let $\tilde{\mathbf{v}}_{k,l} := (\mathbf{v}(\mathbf{x}) \star (\mathbf{e}_1\mathbf{x})^k(\mathbf{x}\mathbf{e}_1)^l)(0)$ and $\tilde{\mathbf{u}}_{k,l} := (\mathbf{u}(\mathbf{x}) \star (\mathbf{e}_1\mathbf{x})^k(\mathbf{x}\mathbf{e}_1)^l)(0)$ differ from zero. Then the misalignment α can be calculated from*

$$\alpha = \frac{\arg(\tilde{\mathbf{u}}_{k,l}\tilde{\mathbf{v}}_{k,l})}{l-k-1}. \tag{5.2}$$

Proof. We look at

$$\begin{aligned}
\tilde{\mathbf{v}}_{k,l} =& (\mathbf{v}(\mathbf{x}) \star \overline{(\mathbf{e}_1\mathbf{x})^k(\mathbf{x}\mathbf{e}_1)^l)(0)} \\
=& \int_{\mathbb{R}^2} \sum_{k',l'=0}^{\infty} (\mathbf{e}_1\mathbf{x})^{k'}(\mathbf{x}\mathbf{e}_1)^{l'}\mathbf{v}_{k',l'}(\mathbf{e}_1\mathbf{x})^k(\mathbf{x}\mathbf{e}_1)^l \,\mathrm{d}\mathbf{x} \\
=& \int_{\mathbb{R}^2} \sum_{k',l'=0}^{\infty} (\mathbf{e}_1\mathbf{x})^{k'}(\mathbf{x}\mathbf{e}_1)^{l'}\mathbf{v}_{k',l'}(\mathbf{e}_1\mathbf{x})^k(\mathbf{x}\mathbf{e}_1)^l \,\mathrm{d}\mathbf{x} \\
=& \int_{\mathbb{R}^2} \sum_{k',l'=0}^{\infty} \mathbf{v}_{k',l'}(\mathbf{x}\mathbf{e}_1)^{k'}(\mathbf{e}_1\mathbf{x})^{l'}(\mathbf{e}_1\mathbf{x})^k(\mathbf{x}\mathbf{e}_1)^l \,\mathrm{d}\mathbf{x} \\
=& \sum_{k',l'=0}^{\infty} \mathbf{v}_{k',l'} \int_{\mathbb{R}^2} (\mathbf{e}_1\mathbf{x})^{k+l'}(\mathbf{x}\mathbf{e}_1)^{l+k'} \,\mathrm{d}\mathbf{x}
\end{aligned} \tag{5.3}$$

and use the polar representation

$$\begin{aligned}
\mathbf{e}_1\mathbf{x} =& |\mathbf{x}| e^{\angle(\mathbf{e}_1,\mathbf{x})\mathbf{e}_{12}} = re^{\varphi \mathbf{e}_{12}}, \\
\mathbf{x}\mathbf{e}_1 =& |\mathbf{x}| e^{\angle(\mathbf{x},\mathbf{e}_1)\mathbf{e}_{12}} = re^{-\varphi \mathbf{e}_{12}},
\end{aligned} \tag{5.4}$$

to show

$$\begin{aligned}
& \int_{\mathbb{R}^2} (\mathbf{e}_1\mathbf{x})^{k+l'}(\mathbf{x}\mathbf{e}_1)^{l+k'} \,\mathrm{d}\mathbf{x} \\
=& \int_0^\infty \int_0^{2\pi} (re^{\varphi \mathbf{e}_{12}})^{k+l'}(re^{-\varphi \mathbf{e}_{12}})^{l+k'} r \,\mathrm{d}\varphi \,\mathrm{d}r \\
=& \int_0^\infty r^{k+k'+l+l'+1} \,\mathrm{d}r \int_0^{2\pi} e^{i\varphi((k+l')-(l+k'))} \,\mathrm{d}\varphi \\
=& 2\pi \int_0^\infty r^{k+l'+l+k'+1} \,\mathrm{d}r \,\delta_{k+l'-l-k'}
\end{aligned} \tag{5.5}$$

To be correct a limit to infinity and a normalization would have to be added, but to keep things short we will assume that the remaining integrals are bounded. Application to (5.3) leads to

$$\begin{aligned}
\tilde{\mathbf{v}}_{k,l} =& \sum_{k',l'=0}^{\infty} \mathbf{v}_{k',l'} 2\pi \int_0^\infty r^{k+l'+l+k'+1} \,\mathrm{d}r \,\delta_{k+l'-l-k'} \\
=& \sum_{l'=0}^{\infty} 2\pi \int_0^\infty r^{2(k+l')} r \,\mathrm{d}r \,\mathbf{v}_{k-l+l',l'} \\
=& \sum_{l'=0}^{\infty} ||\mathbf{x}^{k+l'}||_{L^2}^2 \mathbf{v}_{k-l+l',l'} \\
\overset{j=l'-l}{=}& \sum_{j=-l}^{\infty} ||\mathbf{x}^{k+l'}||_{L^2}^2 \mathbf{v}_{k+j,l+j}.
\end{aligned} \tag{5.6}$$

Taking into account $\mathbf{u}_{k,l} = e^{\alpha \mathbf{e}_{12}(k-l-1)} \mathbf{v}_{k,l}$ we get

$$\begin{aligned}
\tilde{\mathbf{u}}_{k,l} =& \sum_{j=-l}^{\infty} ||\mathbf{x}^{k+l+j}||_{L^2}^2 \mathbf{u}_{l+j,k+j} \\
=& \sum_{j=-l}^{\infty} ||\mathbf{x}^{k+l+j}||_{L^2}^2 e^{\alpha \mathbf{e}_{12}((l+j)-(k+j)-1)} \mathbf{v}_{l+j,k+j} \\
=& e^{\alpha \mathbf{e}_{12}(l-k-1)} \sum_{j=-l}^{\infty} ||\mathbf{x}^{k+l+j}||_{L^2}^2 \mathbf{v}_{l+j,k+j} \\
=& e^{\alpha \mathbf{e}_{12}(l-k-1)} \tilde{\mathbf{v}}_{k,l},
\end{aligned} \tag{5.7}$$

what leads to the assertion. □

Remark 5.2. An orthogonal basis would be more efficient. But since we plan on using only very few of the monomials, the redundancy can be neglected.

Example. The saddle $\mathbf{a}(\mathbf{x})$ from the linear examples has the geometric representation

$$\mathbf{a}(\mathbf{x}) = \mathbf{e}_1\mathbf{x}\mathbf{e}_1, \tag{5.8}$$

that means all new geometric coefficients $\mathbf{a}_{k,l}$ vanish except for

$$\mathbf{a}_{1,0} = \mathbf{e}_1. \tag{5.9}$$

The coefficients in the new geometric representation of its rotated copy $\mathbf{u}(\mathbf{x}) = R_\alpha(\mathbf{a}(R_\alpha^{-1}(\mathbf{x})))$ suffice

$$\mathbf{u}_{k,l} = e^{\alpha \mathbf{e}_{12}(l-k-1)} \mathbf{a}_{k,l}, \tag{5.10}$$

therefore

$$\mathbf{u}_{k,l} = \begin{cases} e^{\alpha \mathbf{e}_{12}(0-1-1)}\mathbf{e}_1, & \text{if } k=1, l=0 \\ 0, & \text{else} \end{cases} \tag{5.11}$$

and we can calculate the misalignment from

$$\alpha = \frac{\arg(\mathbf{u}_{1,0}\mathbf{a}_{1,0})}{-2}, \tag{5.12}$$

compare Theorem 4.2 The correlation of the saddle with $\mathbf{e}_1\mathbf{x}$ yields

$$\begin{aligned} \tilde{\mathbf{a}}_{1,0} =& (\mathbf{a}(\mathbf{x}) \star \mathbf{e}_1\mathbf{x})(0) \\ =& \int_{\mathbb{R}^2} \mathbf{e}_1\mathbf{x}\mathbf{e}_1\mathbf{e}_1\mathbf{x}\, \mathrm{d}\mathbf{x} \\ =& \mathbf{e}_1 \int_{\mathbb{R}^2} \mathbf{x}^2\, \mathrm{d}\mathbf{x} \\ =& \mathbf{e}_1 ||\mathbf{x}||_{L^2}^2 \end{aligned} \tag{5.13}$$

and since its rotated copy has the shape

$$\begin{aligned} \mathbf{u}(\mathbf{x}) =& R_\alpha(\mathbf{a}(R_\alpha^{-1}(\mathbf{x}))) \\ =& e^{-\alpha \mathbf{e}_{12}}\mathbf{e}_1 e^{\alpha \mathbf{e}_{12}}\mathbf{x}\mathbf{e}_1 \\ =& e^{-2\alpha \mathbf{e}_{12}}\mathbf{e}_1\mathbf{x}\mathbf{e}_1 \end{aligned} \tag{5.14}$$

so the correlation coefficient $\tilde{\mathbf{u}}_{1,0}$ suffices

$$\begin{aligned} \tilde{\mathbf{u}}_{1,0} =& (\mathbf{u}(\mathbf{x}) \star \mathbf{e}_1\mathbf{x})(0) \\ =& \int_{\mathbb{R}^2} e^{-2\alpha \mathbf{e}_{12}}\mathbf{e}_1\mathbf{x}\mathbf{e}_1\mathbf{e}_1\mathbf{x}\, \mathrm{d}\mathbf{x} \\ =& e^{-2\alpha \mathbf{e}_{12}}\mathbf{e}_1 ||\mathbf{x}||_{L^2}^2 \end{aligned} \tag{5.15}$$

and we can also calculate the misalignment from

$$\alpha = \frac{\arg(\tilde{\mathbf{u}}_{1,0}\tilde{\mathbf{a}}_{1,0})}{-2}, \tag{5.16}$$

compare Theorem 5.1, what coincides with Remark 3.9.

6. CONCLUSIONS AND OUTLOOK

The geometric cross correlation of two vector fields is scalar and bivector valued. In Theorem 3.8 we learned that iterative application of the encoded rotation completely erases the misalignment of the rotationally misaligned vector fields, if $||\mathbf{v}_1(\mathbf{x})|| \neq 0 \neq ||\mathbf{v}_2(\mathbf{x})||$. These exceptions could also be treated in Algorithm 1. We implemented it and experimentally confirmed the theoretic results.

For the treatment of a more general class of vector fields we suggested the expansion with respect to the geometric basis. We showed, that the rotational misalignment can be detected from geometric correlation with the functions of the geometric basis for any analytic two-dimensional vector field.

Currently we analyze the application of this approach to total rotations of three-dimensional vector fields and look for orthogonal bases, that also have pleasant properties with respect to total rotation.

REFERENCES

1. Lisa Gottesfeld Brown. A survey of image registration techniques. *ACM Computing Surveys*, 24:325–376, 1992.
2. Roxana Bujack, Gerik Scheuermann, and Eckhard Hitzer. Detection of Outer Rotations on 3D-Vector Fields with Iterative Geometric Correlation. *5th conference on Applied Geometric Algebras in Computer Science and Engineering*, 2012.
3. Brian Cabral and Leith Casey Leedom. Imaging vector fields using line integral convolution. In *Proceedings of the 20th annual conference on Computer graphics and interactive techniques*, SIGGRAPH '93, pages 263–270, New York, NY, USA, 1993. ACM.
4. William Kingdon Clifford. Applications of Grassmann's Extensive Algebra. *American Journal of Mathematics*, 1(4):350–358, 1878.
5. J. Ebling and G. Scheuermann. Clifford convolution and pattern matching on vector fields. In *Visualization, VIS 2003. IEEE*, pages 193–200, 2003.
6. Julia Ebling. *Visualization and Analysis of Flow Fields using Clifford Convolution*. PhD thesis, University of Leipzig, Germany, 2006.
7. David Hestenes and Garret Sobczyk. *Clifford Algebra to Geometric Calculus*. D. Reidel Publishing Group, Dordrecht, Netherlands, 1984.
8. Wei Liu and Eraldo Ribeiro. Scale and rotation invariant detection of singular patterns in vector flow fields. In *IAPR International Workshop on Structural Syntactic Pattern Recognition (S-SSPR)*, 2010.
9. C. Eddie Moxey, Todd A. Ell, and Steven J. Sangwine. Vector correlation of color images. *1st European Conference on Color in Graphics, Imaging and Vision, Society for Imaging Science and Technology*, 2002.
10. C. Eddie Moxey, Steven J. Sangwine, and Todd A. Ell. Hypercomplex Correlation Techniques for Vector Images. *Signal Processing, IEEE Transactions*, 51(7):1941–1953, 2003.
11. Azriel Rosenfeld and Avinash C. Kak. *Digital Picture Processing*. Academic Press, Inc., Orlando, FL, USA, 2nd edition, 1982.
12. Gerik Scheuermann. *Topological Vector Field Visualization with Clifford Algebra*. PhD thesis, University of Kaiserslautern, Germany, 1999.
13. Barbara Zitová and Jan Flusser. Image registration methods: a survey. *Image and Vision Computing*, 21(11):977–1000, 2003.